Journal of International Education and Practice

https://ojs.bilpublishing.com/index.php/jiep

ARTICLE

Comparative Study on Results of Euler, Improved Euler and Runge-Kutta Methods for Solving the Engineering Unknown Problems

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ARTICLE INFO

Article history
Issue 3 December 2020
Received: 3 February 2021
Accepted: 9 February 2021
Published: 25 February 2021

Keywords:
Numerical Method
Euler Method
Improved Euler Method
Runge-Kutta Method
Solving the Engineering Problems

ABSTRACT

The paper presents the comparative study on numerical methods of Euler method, Improved Euler method and fourth-order Runge-Kutta method for solving the engineering problems and applications. The three proposed methods are quite efficient and practically well suited for solving the unknown engineering problems. This paper aims to enhance the teaching and learning quality of teachers and students for various levels. At each point of the interval, the value of y is calculated and compared with its exact value at that point. The next interesting point is the observation of error from those methods. Error in the value of y is the difference between calculated and exact value. A mathematical equation which relates various functions with its derivatives is known as a differential equation. It is a popular field of mathematics because of its application to real-world problems. To calculate the exact values, the approximate values and the errors, the numerical tool such as MATLAB is appropriate for observing the results. This paper mainly concentrates on identifying the method which provides more accurate results. Then the analytical results and calculates their corresponding error were compared in details. The minimum error directly reflected to realize the best method from different numerical methods. According to the analyses from those three approaches, we observed that only the error is nominal for the fourth-order Runge-Kutta method.

1. Introduction

The unknown engineering problems came from various points of view. The numerical methods play a vital role in solving the problems with a better experience in science and engineering areas. Differential equations solve engineering mathematical problems in more or less every section of science, technology, engineering, and mathematics (STEM). In engineering mathematics section, several real problems get up in the form of differential equations. These equations are either in the form of an ordinary differential equation or partial differential equation for defining the unknown problems. Customarily, most of the unknown engineering problems modelled by these differential equations are accordingly complicated that it is inflexible to determine the exact solution. One of the three approaches is occupied to imprecise the solution. In the methods of Euler, Improved Euler and Runge-Kutta, the interval length h should be kept back small, and these methods can be applied for tabularizing y over bounded

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 DOI: https://doi.org/10.30564/jiep.v3i3.2881

range only. The original conditions are quantified at the initial point only are entitled initial value problem[5, 6].

The analytical approaches can be applied to elucidate only a designated class of differential equations that manage physical coordination do not retain, on the whole, closed-form solutions[3]. The numerical methods change to the approximate results. In this paper, the three standard numerical methods such as Euler, Improved Euler, and Runge-Kutta to elucidate initial value problems of differential equations are presented[1, 8]. In contrast, the Runge-Kutta method's results congregate closer to analytical solutions, and its requirements less iteration to contribute precise solutions. Consequently, the Runge-Kutta method provides better results for solving unknown engineering problems.

The paper is well-organized with the following sections. Section II presents the background knowledge on three techniques. Section III mentions analysis and discussion. Section IV gives the comparison results and errors. Finally, section V offers the conclusion of the study.

2. Background Knowledge on Three Numerical Methods

A. *Euler Method*

The Euler method is the unpretentious one-step method to solve the unknown engineering problem[7]. It is an uncomplicated unequivocal technique for numerical ordinary differential equations. It is the primary numerical method for solving initial value problems and exemplifies the perceptions convoluted in the advanced methods[1]. It is indispensable to study for the reason that the error analysis is easier to apprehend. The all-purpose formula for Euler approximation [5-6] is

$$y_{n+1} = y_n + hf(x_n, y_n), \ n = 0,1,2,... \quad (1)$$

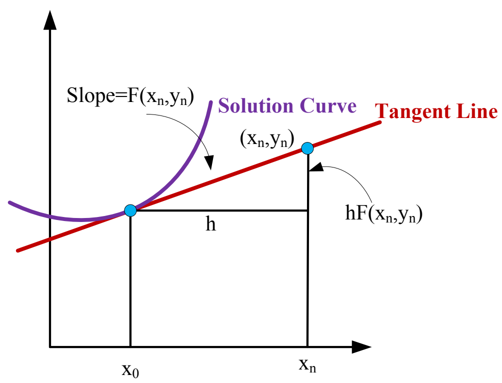

Figure 1. Illustration of the Euler Method

B. *Improved Euler Method*

The improved Euler technique stretches a superior enlargement in accurateness over the original Euler method. This technique is entitled after Karl Heun[1]. In this technique, two derivatives are utilized to achieve an improved estimation of the slope for the all-inclusive interval by averaging them. This technique is established on two values of the dependent variables the predicted values y^*_{n+1} and the final value y_{n+1} [5-6] which are given by

$$y^*_{n+1} = y_n + hf(x_n, y_n) \quad (2)$$

The broad-spectrum formulation for Improved Euler approximation is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y^*_{n+1}\right)\right], \ n = 0,1,2,... \quad (3)$$

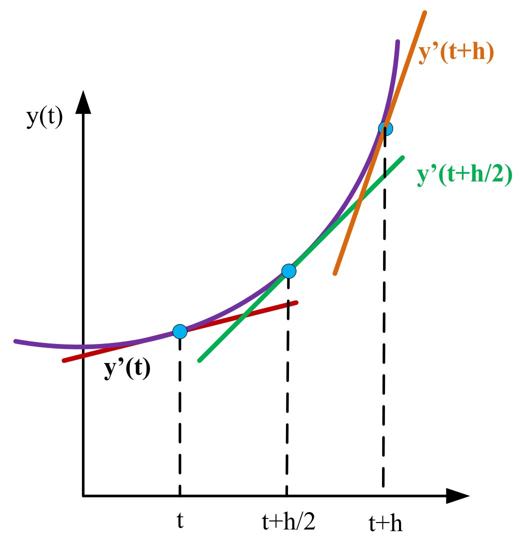

Figure 2. Illustration of the Improved Euler Method

C. *Runge-Kutta Method*

Two German mathematicians concocted this technique. The Runge-Kutta technique is the greatest widespread technique because it is somewhat precise, stable and tranquil to create the mathematical program for numerical analyses. This technique can elucidate Taylor's series solution without difficulty. It does not ultimatum the aforementioned computational of higher $y(x)$ derivatives in Taylor's series technique[4]. The fourth-order Runge-Kutta method is extensively utilized for unravelling initial value problems for the ordinary differential equation. The broad-spectrum formulation for Runge-Kutta approximation [5-6] is

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right], \ n = 0,1,2,... \quad (4)$$

DOI: https://doi.org/10.30564/jiep.v3i3.2881

$$k_1 = hf\left(x_n, y_n\right) \tag{5}$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \tag{6}$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \tag{7}$$

$$k_4 = hf\left(x_n + h, y_n + k_3\right) \tag{8}$$

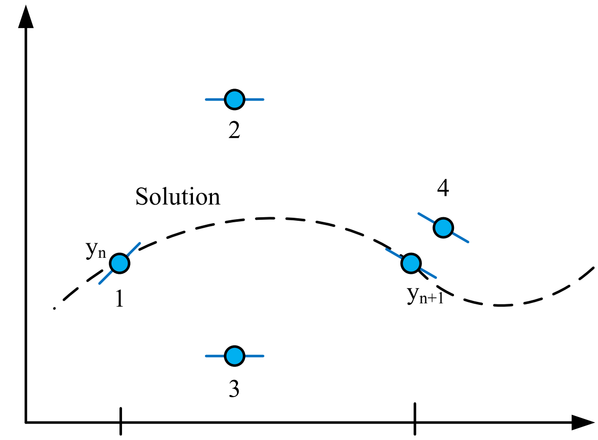

Figure 3. Fourth Order Runge-Kutta Method

D. *Errors Calculation*

The calculation of Error-values is essential to check the performance of the applied technique. The measurement of errors could be calculated as follows:

$$\text{Actual Error} = E_{TRUE} - E_{CALC} \tag{9}$$

$$\text{Absolute Error} = \left|E_{TRUE} - E_{CALC}\right| \tag{10}$$

$$\text{Relative Error} = \frac{\left|E_{TRUE} - E_{CALC}\right|}{\left|E_{TRUE}\right|} \tag{11}$$

$$\text{Percentage Error} = \frac{\left|E_{TRUE} - E_{CALC}\right|}{\left|E_{TRUE}\right|} \times 100\% \tag{12}$$

3. Analysis and Discussions

This section mentions the analyses and discussions on the three types of numerical methods. In this study, there are three case studies for checking the performance of the numerical techniques.

3.1 Case Study (1)

In this case, we focused on the first case of $y' - y = e^{2x}$ numerical methods to find the performance results with the help of MATLB[2,9,10] language. The proposed engineering problem in this case study 1 is significant to solve the control system design and engineering scenario. For this analysis, some parameters are assumed for performance checking $y(0) = 1$ and $h = 0.1$: our initial condition and specific constant parameter with 10 steps.

After implementing the numerical calculation with MATLB[2,9,10], the exact values can be observed as the array style with Exact = [1.0000 1.2214 1.4918 1.8221 2.2255 2.7183 3.3201 4.0552 4.9530 6.0496 7.3891] and the values of Euler =[0 1.0000 0.6000 3.0471 0.1000 1.2000 0.7000 3.6838 0.2000 1.4421 0.8000 4.4577 0.3000 1.7355 0.9000 5.3988 0.4000 2.0913 1.0000 6.5437 0.5000 2.5230]. If we choose the value of exact is7.3891 and the value of approximate is 6.5437, the actual error value may be obtained as 0.8454.

If we choose the mode of Improved Euler techniques, the actual error value may be obtained as 0.0134. If we opt for the technique of Runge-Kutta, the actual error value could be observed as 0.0001.

3.2 Case Study (2)

In this case, we focused on the first case of $y' + 2y = 4\cos 2x$ numerical methods to find the performance results with the help of MATLB[2,9,10] language. The proposed engineering problem in this case study 2 is significant to solve the dynamical system design and engineering picture. For this analysis, some parameters are assumed for performance checking $y(0) = 2$ and $h = 0.1$: our initial condition and specific constant parameter with 10 steps.

After implementing the numerical calculation with MATLB[2,9,10], the exact values can be observed as the array style with Exact = [2.0000 1.9975 1.9808 1.9388 1.8634 1.7497 1.5956 1.4020 1.1723 0.9119 0.6285] and the values of Euler = [0 2.0000 0.6000 1.6549 0.1000 2.0000 0.7000 1.4689 0.2000 1.9920 0.8000 1.2431 0.3000 1.9620 0.9000 0.9828 0.4000 1.8998 1.0000 0.6954 0.5000 1.7985]. If we choose the value of exact is 0.6285 and the value of approximate is 0.6954, the actual error value may be obtained as -0.0669.

If we choose the mode of Improved Euler techniques, the actual error value may be obtained as 0.0062. If we opt for the technique of Runge-Kutta, the actual error value could be observed as 0.

3.3 Case Study (3)

In this case, we focused on the first case of $y' = (y - x)^2$

DOI: https://doi.org/10.30564/jiep.v3i3.2881

numerical methods to find the performance results with the help of MATLB[2,9,10] language. The proposed engineering problem in this case study 3 is significant to solve the vibrational system design and engineering depiction. For this analysis, some parameters are assumed for performance checking $y(0)=0$ and $h=0.1$: our initial condition and specific constant parameter with 10 steps.

After implementing the numerical calculation with MATLB[2,9,10], the exact values can be observed as the array style with Exact = [0 0.0003 0.0026 0.0087 0.0201 0.0379 0.0630 0.0956 0.1360 0.1837 0.2384] and the values of Euler = [0 0 0.6000 0.0508 0.1000 0 0.7000 0.0810 0.2000 0.0010 0.8000 0.1193 0.3000 0.0050 0.9000 0.1656 0.4000 0.0137 1.0000 0.2196 0.5000 0.0286]. If we choose the value of exact is 0.2384 and the value of approximate is 0.2196, the actual error value may be obtained as 0.0188.

If we choose the mode of Improved Euler techniques, the actual error value may be obtained as -0.0013. If we opt for the technique of Runge-Kutta, the actual error value could be observed as 0.

4. Comparison of Results And Errors

In this section, the author compares numerical results with the exact solution to determine corresponding errors and check out which methods provide better results.

E. *Comparison Results and Errors of Case Study (1)*

Table 1. Comparison Results of Case Study (1)

n	x_n	Exact value	Euler method	Improved Euler method	Runge-Kutta method
0	0	1.0000	1.0000	1.0000	1.0000
1	0.1	1.2214	1.2000	1.2211	1.2214
2	0.2	1.4918	1.4421	1.4911	1.4918
3	0.3	1.8221	1.7355	1.8208	1.8221
4	0.4	2.2255	2.0913	2.2234	2.2255
5	0.5	2.7183	2.5230	2.7152	2.7183
6	0.6	3.3201	3.0471	3.3158	3.3201
7	0.7	4.0552	3.6838	4.0494	4.0552
8	0.8	4.9530	4.4577	4.9452	4.9530
9	0.9	6.0496	5.3988	6.0394	6.0496
10	1	7.3891	6.5437	7.3757	7.3890

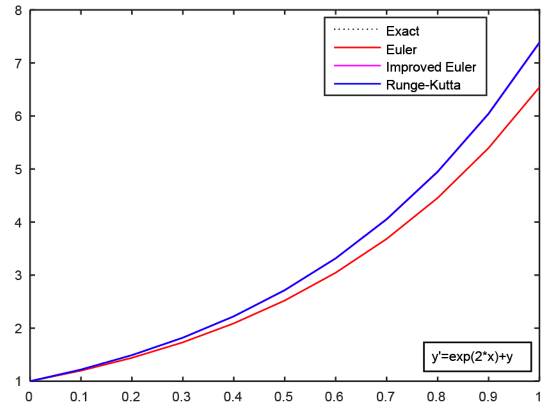

Figure 4. Performance Comparison Results for Case Study (1)

Table 2. Comparison Errors of Case Study (1)

Euler method	Improved Euler method	Runge-Kutta Method
0.8454	0.0134	0.0001

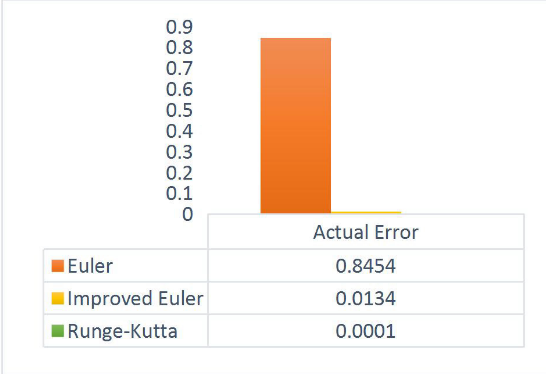

Figure 5. Comparison Errors of Case Study (1)

F. *Comparison Results and Errors of Case Study (2)*

Table 3. comparison results of Case Study(2)

n	x_n	Exact value	Euler method	Improved Euler method	Runge-Kutta method
0	0	2.0000	2.0000	2.0000	2.0000
1	0.1	1.9975	2.0000	1.9960	1.9975
2	0.2	1.9808	1.9920	1.9778	1.9808
3	0.3	1.9388	1.9620	1.9342	1.9388
4	0.4	1.8634	1.8998	1.8574	1.8634
5	0.5	1.7497	1.7985	1.7426	1.7496
6	0.6	1.5956	1.6549	1.5879	1.5956
7	0.7	1.4020	1.4689	1.3940	1.4020
8	0.8	1.1723	1.2431	1.1645	1.1723
9	0.9	0.9119	0.9828	0.9047	0.9119
10	1	0.6285	0.6954	0.6223	0.6285

DOI: https://doi.org/10.30564/jiep.v3i3.2881

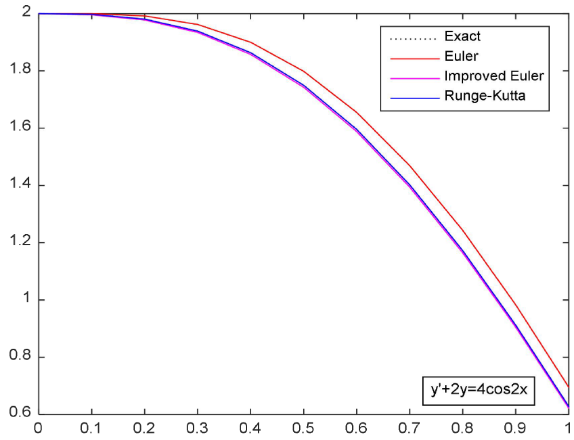

Figure 6. Comparison Errors of Case Study (2)

Table 4. Comparison Errors of Case Study (2)

Euler method	Improved Euler method	Runge-Kutt Method
-0.0669	0.0062	0.0000

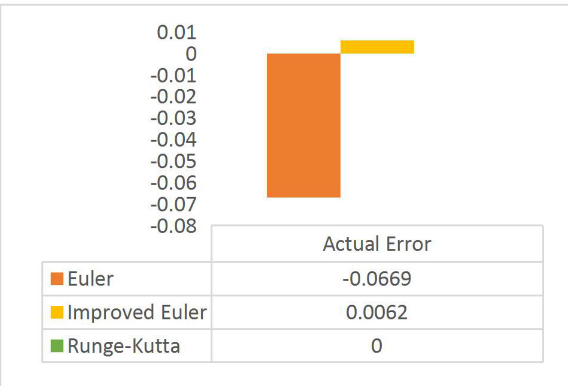

Figure 7. Comparison Errors of Case Study (2)

G. *Comparison Results and Errors of Case Study (3)*

Table 5. Comparison Results of Case Study (3)

n	x_n	Exact value	Euler method	Improved Euler method	Runge-Kutta method
0	0	0.0000	0.0000	0.0000	0.0000
1	0.1	0.0003	0.0000	0.0005	0.0003
2	0.2	0.0026	0.0010	0.0030	0.0026
3	0.3	0.0087	0.0050	0.0092	0.0087
4	0.4	0.0201	0.0137	0.0207	0.0201
5	0.5	0.0379	0.0286	0.0387	0.0379
6	0.6	0.0630	0.0508	0.0640	0.0630
7	0.7	0.0956	0.0810	0.0968	0.0956
8	0.8	0.1360	0.1193	0.1372	0.1360
9	0.9	0.1837	0.1656	0.1850	0.1837
10	1	0.2384	0.2196	0.2397	0.2384

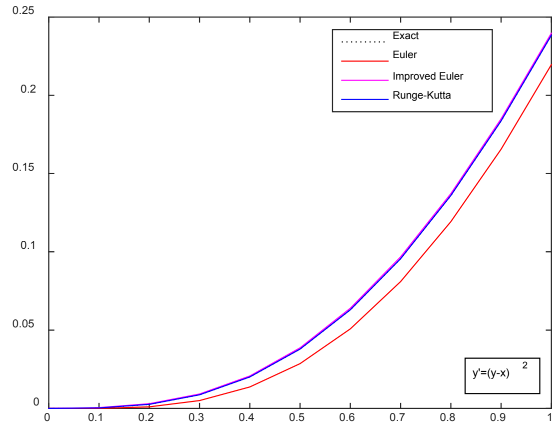

Figure 8. Comparison Errors of Case Study (3)

Table 6. Comparison Errors of Case Study (3)

Euler method	Improved Euler method	Runge-Kutta Method
0.0188	-0.0013	0.0000

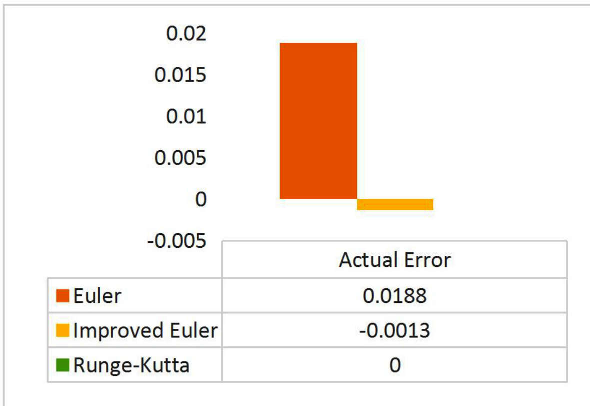

Figure 9. Comparison Errors of Case Study (3)

Upon spread over the Runge-Kutta technique, there is no evidence of any error values. This is because only four significant digits are well-thought-out. If more than four significant digits are utilized in that analyses, there may be the error values.

From the directly above tables and figures, we could observe that the values from the Runge-Kutta technique's result are very close to the exact value. This technique could provide high-level precise results than the Euler technique and the Improved Euler technique. Above three case studies, we could perceive that the Runge-Kutta technique stretches the minimum error values. Therefore, the Runge-Kutta technique is superior to the other two approaches.

5. Conclusions

In this paper, the Euler technique, Improved Euler technique, and the Runge-Kutta technique solve ordinary differential equations in initial value problems for different kinds of engineering unknown problems. Finding more precise results needs Runge-Kutta for all approaches. The numerical solutions achieved by the three proposed approaches are a good covenant with the exact solution. Compared with the three methods under investigation, the convergence rate of Euler, Improved Euler and Runge-Kutta approaches could also be observed. The Euler technique and the Improved Euler technique were less precise values due to the imprecise numerical results acquired from the approximate solution compared to the exact solution. By and large, the Runge- Kutta technique is more precise than the other two techniques. The technique can help to study differential equations that have wide applications in daily life such as control system, dynamical system, and vibrational system design and engineering. The technique converges faster to the exact solution than the Euler technique, Improved Euler technique. It might be established that the Runge-Kutta technique is efficient and effective with good accuracy for solving the unknown engineering problems.

Acknowledgement

I am grateful to Dr. Myat Shwe Wah, Professor, Head of Department of Physics, Myanmar Aerospace Engineering University and Dr. Tun Ko Ko, Professor, Head of Department of Engineering Mathematics, Myanmar Aerospace Engineering University for their support and general guidance throughout the development of this research paper. I would like to express my heartfelt appreciation to Daw Su Su Latt, Associate Professor, Faculty of Language (English), University of Computer Studies (Thaton). She has provided a huge amount of her precious time and effort to edit my paper from the language point of view.

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DOI: https://doi.org/10.30564/jiep.v3i3.2881